Journal of Applied Mechanics and Technical Physics, Vol. 41, No. 6, 2000

FREE OSCILLATIONS OF A THIN FLUID LAYER OF FINITE ELECTRIC CONDUCTIVITY UNDER THE ACTION OF AN EXTERNAL MAGNETIC FIELD

A. I. Zadorozhnyi and R. A. Gruntfest

UDC 532.59:533.951

The problem of free oscillations of a thin layer of a heavy, incompressible, inviscid fluid of finite electrical conductivity in a horizontal magnetic field is reduced to a system of integrodifferential Fredholm equations with variable coefficients. A numerical analysis is performed over a broad range of input parameters, and the results obtained are supplemented with asymptotic formulas with large and small magnetic Reynolds numbers. A classification of the resulting wave modes is proposed. It is shown that certain conditions can lead to the occurrence of unstable oscillations of the fluid layer that grow in time.

A magnetic field has a significant effect on the free oscillations of a fluid of finite electrical conductivity, primarily because of dissipation of kinetic energy into Joule heat. A review of the literature shows that this problem was the subject of only a few studies (see, e.g., [1–3]). In the fluid dynamics of wave motion, it is usual to employ two approaches — long-wave and short-wave approximations [4]. The short-wave approximation uses the model of a fluid of infinite depth, and the long-wave approximation is based on the model of a thin layer. The present work is devoted to a theoretical and numerical study of the spectrum of free oscillations in the thin-layer approximation. We consider surface and internal MHD waves for a heavy, inviscid fluid of finite electrical conductivity under the action of an external magnetic field induced by a direct electric current. Calculation results are given, and a classification of the wave modes is proposed.

1. Formulation of the Problem. We consider the physical model shown in Fig. 1 (the z axis is directed vertically upward). In the lower half-plane $-\infty < z < -d$ there is a substrate (solid dielectric). Vacuum or a nonionized gas is located in the upper half-plane $0 < z < \infty$. These regions are subjected to the action of a horizontal magnetic field of constant intensity H_i ($i = 1$ for vacuum and 2 for the substrate). The layer $-d < z < 0$ is occupied by a conducting inviscid fluid. Let the fluid be acted upon by gravity and magnetic field $H(z)$ directed along the x axis. This magnetic field can be generated by an electric current passing through the fluid perpendicular to the Oxy plane. According to Ampere's law, the density of this current is $j = dH/dz$ (in the SI-system). If the current density is constant over the depth of the layer, the magnetic intensity depends linearly on the depth $H(z) = H_0(1 - \gamma z/d)$, where H_0 is the intensity on the free surface of the fluid and $\gamma = -jd/H_0$. Assuming that the fluid flow is planar we introduce the velocity vector $\boldsymbol{V} = (V_x, V_z)$ and the perturbation of the magnetic intensity $\boldsymbol{h} = (h_x, h_z)$ due to the fluid flow. Since below we consider natural oscillations of the fluid, we introduce the wavelength L, which is used as the characteristic dimension along the x axis. The parameter $\delta = d/L$ is considered small, which corresponds to the thin-layer approximation. Within the framework of this approximation, the equations of motion of the conducting fluid in the dimensionless variables take the form [2, 4]

Rostov State University, Rostov-on-Don 344006. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 41, No. 6, pp. 27–35, November–December, 2000. Original article submitted February 9, 1999; revision submitted March 9, 2000.

 0021-8944/00/4106-0982 $25.00

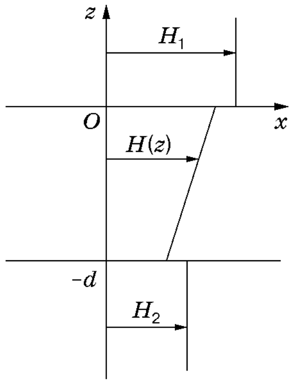

Fig. 1. Three-layer model “vacuum–fluid–substrate.”

$$\frac{\partial V_x}{\partial t} = -\frac{\partial p^*}{\partial x} + \mathrm{Al}\Big(H'(z)h_z - H(z)\,\frac{\partial h_z}{\partial z}\Big), \quad \frac{\partial p^*}{\partial z} = 0, \quad \frac{\partial V_x}{\partial x} + \frac{\partial V_z}{\partial z} = 0, \tag{1}$$

where $p^* = p + z + \mathrm{Al}\,H^2(z)/2 + \mathrm{Al}H(z)h_x$ (p is the fluid-dynamic pressure). The magnetic intensity satisfies the induction and continuity equations

$$\frac{\partial h_x}{\partial t} = -\frac{\partial}{\partial z}\,(H(z)V_z) + \frac{1}{\mathrm{Re}_m}\,\frac{\partial^2 h_x}{\partial z^2},$$
$$\frac{\partial h_z}{\partial t} = \frac{\partial}{\partial x}\,(H(z)V_z) + \frac{1}{\mathrm{Re}_m}\,\frac{\partial^2 h_z}{\partial z^2}, \qquad \frac{\partial h_x}{\partial x} + \frac{\partial h_z}{\partial z} = 0. \tag{2}$$

In (1) and (2), $\mathrm{Al} = \mu_c H_0^2/(\rho g d)$ is the Alfén number, $\mathrm{Re}_m = d^2\mu_c\sigma\sqrt{gd}/L$ is the magnetic Reynolds number, ρ, σ, and μ_c are the density, electric conductivity, and magnetic permeability of the fluid, g is the free-fall acceleration, and d is the thickness of the layer. The magnetic field directed by the fluid flow in the regions of vacuum and substrate satisfy the Maxwell equations [2]

$$\frac{\partial^2 h_{ix}}{\partial x^2} + \delta^2\,\frac{\partial^2 h_{ix}}{\partial z^2} = 0, \qquad \frac{\partial h_{ix}}{\partial x} + \frac{\partial h_{iz}}{\partial z} = 0 \qquad (i = 1,\, 2). \tag{3}$$

We consider the boundary conditions. On the fluid–substrate interface, the vertical velocity of the fluid is equal to zero and the normal projection of the magnetic-induction vector and tangential component of the magnetic-stress tensor are continuous:

$$V_z = 0, \quad \mu h_z = h_{2z}, \quad \mu = \mu_c/\mu_2, \quad \mu(1+\gamma+h_x)h_z = (k_2 + h_{2x})h_{2z}, \quad z = -1.$$

Here $k_2 = H_2/H_0$ and μ_2 is the magnetic permeability of the substrate. For the third condition, the following cases are possible.

1. The normal projections of the magnetic intensity are equal to zero:

$$h_z = 0, \qquad h_{2z} = 0, \qquad z = -1. \tag{4}$$

The tangential projections of the magnetic intensity can have a discontinuity due to the existence of a boundary current in the fluid that can have both constant and variable components. If $k_2 = 1 + \gamma$, a direct current is absent, and in the case $h_{2x} = h_x$, the variable component of the current disappears.

2. The tangential projections of the magnetic intensity are identical:

$$h_x = h_{2x}. \tag{5}$$

We note that free oscillations are possible only for $k_2 = 1 + \gamma$. The equation of the fluid–vacuum interface is written as $z = \zeta(x, y, t)$. Here, the following conditions must be satisfied: the kinematic condition $\partial\zeta/\partial t = V_z$ ($z = 0$), the dynamic continuity condition for the normal component of the total stress tensor

$$-p + T_{nn} = -p_a + T_{1nn} \qquad (z = 0) \tag{6}$$

($p_a = \text{const}$ is the external pressure on the surface), and the continuity condition for the normal projection of the magnetic induction and the tangential component of the magnetic-stress tensor [2]

$$\mu\Big(h_z - \frac{\partial\zeta}{\partial x}\Big) = h_{1z} - k_1\,\frac{\partial\zeta}{\partial x}, \quad \mu = \frac{\mu_c}{\mu_1}, \quad \mu(1+h_x)\Big(h_z - \frac{\partial\zeta}{\partial x}\Big) = (k_1 + h_{1x})\Big(h_{1z} - k_1\,\frac{\partial\zeta}{\partial x}\Big) \quad (z=0).$$

Here $k_1 = H_1/H_0$ and μ_1 is the magnetic permeability of vacuum (values $\mu > 1$ correspond to the magnetized fluid [5]). Again, the following two cases are possible:

Case 1. The normal projections of the magnetic intensity are equal to zero, and

$$\frac{\partial\zeta}{\partial x} = h_x, \qquad k_1\,\frac{\partial\zeta}{\partial x} = h_{1x}; \tag{7}$$

Case 2. The tangential projections of the magnetic intensity are identical:

$$h_x = h_{1x}, \qquad k_1 = 1. \tag{8}$$

After substitution of the expressions for the normal components of the total stress tensor, condition (6) becomes

$$p^* = p_a + \mathrm{Al}/2 + \mathrm{Al}\,h_{1x} + \zeta \qquad (z=0). \tag{9}$$

The boundary conditions should be supplemented with the requirement that the magnetic-field perturbations damp at infinity in the vacuum and substrate regions:

$$h_{ix} = 0, \qquad h_{iz} = 0, \qquad z \to \pm\infty \qquad (i = 1,\, 2). \tag{10}$$

For metallurgical applications, natural conditions are the continuity conditions (5) and (8) for the tangential components of the magnetic intensity on the surface and at the bottom [2], and, hence, they are used in the present calculations. Other combinations of the boundary conditions are of theoretical interest [3].

2. Separation of Variables and Reduction to a Boundary-Value Problem for a System of Integrodifferential Equations. All functions are sought in the form $f(x,z,t) = F(z)\exp(-\lambda t + ix)$. We consider the substrate region ($-\infty < z < -1$), assuming $(h_{2x}, h_{2z}) = (X_2(z), Z_2(z))\exp(-\lambda t + ix)$. Using Eqs. (3) and the damping conditions (10) for $i = 2$, it is easy to show that for small δ (thin fluid layer), we have $X_2(z) = O(\delta) = 0$ and $Z_2(z) = Z_2(-1)\exp(\delta(z+1))$. From this, taking into account boundary condition (4), we obtain

$$h_{2x} = 0, \qquad h_{2z} = 0. \tag{11}$$

Similarly, for vacuum ($0 < z < \infty$), we introduce $(h_{1x}, h_{1z}) = (X_1(z), Z_1(z))\exp(-\lambda t + ix)$ and taking into account (3) and (10), we obtain $X_1(z) = O(\delta) = 0$ and $Z_1(z) = Z_1(0)\exp(-\delta z)$ for $i = 1$. Finally, for the fluid, we introduce the notation $(h_x, h_z, V_x, V_z, \zeta, p) = (X(z), Z(z), U(z), W(z), S, P)\exp(-\lambda t + ix)$ and write the equations of induction and motion (1) and (2) in the form

$$\begin{gathered} -\lambda X(z) = -\frac{\partial}{\partial z}(HW) + \frac{1}{\mathrm{Re}_m}X''(z), \quad -\lambda Z(z) = iH(z)W(z) + \frac{1}{\mathrm{Re}_m}Z''(z), \\ -\lambda U(z) = -iP + \mathrm{Al}\,(ZH' - HZ'), \qquad Z'(z) + iX(z) = 0. \end{gathered} \tag{12}$$

At the bottom ($z = -1$), for case 1 of boundary conditions (4), we have $W(-1) = 0$ and $Z(-1) = 0$, and, hence, from the second of Eqs. (12), it follows that $Z''(-1) = 0$. Differentiating the third of Eqs. (12) with respect to z, and setting $z = -1$, we obtain the condition

$$U'(-1) = 0. \tag{13}$$

For case 2 of the conditions at the bottom (5), the induction equations (12) yield $Z'' = -\mathrm{Re}_m\lambda Z$, $X'' = \mathrm{Re}_m(1+\gamma)W'$, and $z = -1$. With allowance for these equalities, the third of Eqs. (12) gives

$$\lambda U(-1) + \frac{1}{\lambda}\int\limits_{-1}^{0} U(z)\,dz + \frac{\gamma}{\mathrm{Re}_m(1+\gamma)}\,U'(-1) = 0. \tag{14}$$

On the free surface ($z = 0$), for case 1 of conditions (7) we obtain

$$U'(0) = 0. \tag{15}$$

For case 2 [see (8)], the following equalities are valid: $\mu(Z - iS) = Z_1 - iS$, $Z' = 0$, and $W = -\lambda S$ at $z = 0$. From this, taking into account Eqs. (12) we have

$$\lambda U(0) + \frac{1 + \mathrm{Al}\gamma}{\lambda} \int\limits_{-1}^{0} U(z)\, dz + \frac{\gamma}{\mathrm{Re}_m} U'(0) = 0. \tag{16}$$

Elimination of P from the third of Eqs. (12) using (9) gives the equation

$$\lambda U(z) + \frac{1}{\lambda} \int\limits_{-1}^{0} U(\xi)\, d\xi = \mathrm{Al}\, H^2(z) \Big(\frac{Z(z)}{H(z)}\Big)'.$$

We make the replacement $\Phi(z) = (Z(z)/H(z))' = -(1/H(z))(iX(z) + \Gamma(z)Z(z))$, where $\Gamma(z) = H'(z)/H(z)$. The function $\Gamma(z)$ characterizes the vertical stratification of the magnetic field. The second of Eqs. (12) is divided by $H(z)$ and differentiated with respect to z. Taking into account the continuity equation, we obtain $U(z) + \lambda\Phi(z) + (1/\mathrm{Re}_m)(Z''(z)/H(z))' = 0$. Further transformations involve the elimination of $Z(z)$ from the last expression. As a result, for the functions $U(z)$ and $\Phi(z)$, we have the following system of homogeneous integrodifferential equations:

$$\begin{gathered} \lambda U(z) + \frac{1}{\lambda} \int\limits_{-1}^{0} U(\xi)\, d\xi - \mathrm{Al}\, H^2(z)\Phi(z) = 0, \\ \Phi''(z) + 2\Gamma(z)\Phi'(z) - 2\Gamma^2(z)\Phi(z) + \mathrm{Re}_m(U(z) + \lambda\Phi(z)) = 0 \end{gathered} \tag{17}$$

and the boundary conditions on the surface (15) or (16) and at the bottom (13) or (14). In what follows, to eliminate the occurrence of singular points (poles) in the coefficients of system (17), we assume that $\gamma > -1$. It should be noted that the spectral parameter λ appears in (17) in a nonlinear manner.

3. Derivation of the Energy-Balance Equation. The energy-balance equation is commonly used to prove the free oscillation theorem. However, boundary conditions (13)–(16) do not always allow one to do this. Nevertheless, the derivation of the energy-balance equation can be useful for obtaining some estimates. We multiply the third of Eqs. (12) by $\bar{U}(z)$ (complex-conjugate quantity), write the first of Eqs. (12) in complex-conjugate form, and multiply it by $\mathrm{Al}\, X(z)$. Then, integrating both equalities with respect to z in the limits $(-1, 0)$ and combining the integration results, we obtain the relation

$$\lambda\|U\|^2 + \bar{\lambda}\,\mathrm{Al}\|X\|^2 - \frac{\mathrm{Al}}{\mathrm{Re}_m}\|X'\|^2 + \frac{1}{\lambda}|W(0)|^2 - \mathrm{Al}\,\bar{\lambda}\gamma|Z(0)|^2 - \frac{\gamma\lambda}{\mathrm{Re}_m} Z''(0)Z(0) = 0,$$

which, by virtue of the equality $\|X\|^2 = \|H\Phi\|^2 - \gamma|Z(0)|^2$, becomes

$$\lambda\|U\|^2 + \bar{\lambda}\,\mathrm{Al}\,\|H\Phi\|^2 - \frac{\mathrm{Al}}{\mathrm{Re}_m}\|X'\|^2 + \frac{1}{\lambda}|W(0)|^2 - \frac{\gamma\lambda}{\mathrm{Re}_m} U'(0)Z(0) = 0. \tag{18}$$

Thus, the last term outside the integral is absent in the following cases:

1) The external magnetic field in the fluid is constant ($\gamma = 0$);

2) On the free surface, the normal components of the magnetic intensity are equal to zero: $U'(0) = 0$ (case 1). In this case, after separation of the real and imaginary parts, formula (18) leads to the inequality $\mathrm{Re}\,(\lambda) > 0$, which implies that *all free oscillations are damping.* In the remaining cases [variable magnetic field and identical tangential projections of the magnetic intensity on the free surface (case 2)] the extra-integrand term enters in expression (18), which indicates that certain conditions can lead to the occurrence of stable modes. Calculations confirm that they occur for small Re_m.

4. Implementation of Calculations and Analysis of Results. Let us reduce the integro-differential system of equations (17) to a boundary-value problem for a system of ordinary differential equations. We introduce the functions

$$V(z) = iW(z) = \int_{-1}^{z} U(\xi)\,d\xi, \quad \varphi(z) = H^2(z)\Phi(z), \quad \psi(z) = \varphi'(z).$$

Then, system (17) becomes

$$V' = U, \quad U' = \frac{\psi}{\tilde\lambda}, \quad \varphi' = \psi, \quad \psi' = -\frac{2\gamma}{H(z)}\,\psi - \widetilde{\mathrm{Re}}_m\,(\tilde\lambda\varphi + H^2(z)U),$$
$$\widetilde{\mathrm{Re}}_m = \mathrm{Re}_m\sqrt{\mathrm{Al}}, \qquad \lambda = \tilde\lambda\sqrt{\mathrm{Al}}. \tag{19}$$

As the boundary conditions, we have any pair combination of equalities (one at the bottom and one at the free surface)

$$\psi(-1) = 0, \qquad \tilde\lambda U(-1) + \frac{1}{\tilde\lambda\,\mathrm{Al}}\,V(-1) + \frac{\gamma}{\widetilde{\mathrm{Re}}_m\tilde\lambda(1+\gamma)}\,\psi(-1) = 0,$$
$$\psi(0) = 0, \qquad \tilde\lambda U(0) + \frac{1+\mathrm{Al}\,\gamma}{\tilde\lambda\,\mathrm{Al}}\,V(0) + \frac{\gamma}{\widetilde{\mathrm{Re}}_m\tilde\lambda}\,\psi(0) = 0, \tag{20}$$

which should be supplemented with the equalities

$$V(-1) = 0, \qquad \mathrm{Al}\,(\tilde\lambda U(0) - \varphi(0)) + V(0)/\tilde\lambda = 0. \tag{21}$$

The last boundary condition follows from the first of Eqs. (17) at $z = 0$. Below, the tilde sign above the corresponding quantities is omitted. When integrating the boundary-value problem (19)–(21), it is necessary to treat the spectral parameter λ and the sought eigenfunctions as complex quantities and to introduce real and imaginary parts for them. As a result, we obtain a real system of the eighth order and eight boundary conditions. This system with initial conditions at the bottom ($z = -1$) is solved numerically by the Runge–Kutta method of the fifth order with a relative error of order 10^{-6}. Satisfaction of the boundary conditions on the surface ($z = 0$) and the condition for existence of a nontrivial solution give a complex nonlinear system for $\lambda = \alpha + i\beta$, which was solved numerically by the Newton method [6] with a relative error of 10^{-4}. In the calculations, various combinations of the boundary conditions were used. In the present work, as noted above, we chose the continuity conditions for the tangential projections of the magnetic intensity at the bottom and the free surface, which correspond to metallurgical processes. The guaranteed relative error in calculations of simple eigenvalues is equal to 10^{-4}.

The dispersion pattern in the complex plane λ for $\gamma = 0.4$ (the external magnetic field intensity increases from the surface to the bottom) and fixed Alfvén numbers $\mathrm{Al} = 9, 10, \ldots, 14$ is shown in Fig. 2. It is possible to distinguish two oscillation modes that behave differently as $\mathrm{Re}_m \to \infty$.

The first mode (*high-frequency gravitational surface waves*) is characterized by a continuous limiting transition to a fluid of infinite conductivity, whose asymptotic behavior is constructed by the Lomov method [7] and has the form

$$\lambda = i\beta + \frac{1-i}{\sqrt{2\,\mathrm{Re}_m}}\,\beta_1 + O\Big(\frac{1}{\mathrm{Re}_m}\Big), \qquad U(z) = \frac{\beta^2 - (1+\gamma)^2}{\beta^2 + (1-\gamma)^2},$$

$$\beta_1 = \frac{V(0)(U(0) - (1+\gamma\,\mathrm{Al})/(\mathrm{Al}\,\beta^2)V(0) + 1/\sqrt{U(0)}(1 - V(0)/(\mathrm{Al}\,\beta^2)))}{\mathrm{Al}\,\beta\sqrt{\beta - 1/\beta}(\|U\|^2 + \|HU\|^2/\beta^2 + V(0)/(\mathrm{Al}\,\beta^2))}.$$

Here $U(z)$ and β are the eigenfunction and the eigenvalue for the degenerate problem ($\mathrm{Re}_m = \infty$) and β is the larger root of the dispersion equation

$$\Big|\frac{(\beta-1)(\beta+1+\gamma)}{(\beta+1)(\beta-1-\gamma)}\Big| = \exp(2\,\mathrm{Al}\,\beta\gamma), \qquad \beta > \max(1.1+\gamma), \qquad \mathrm{Al}\,|\gamma| < 1. \tag{22}$$

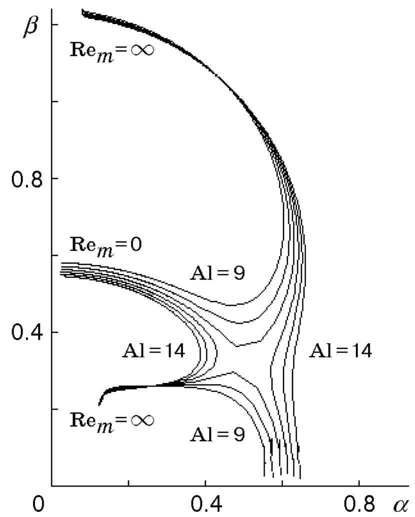

Fig. 2. Dispersion curves in the plane (α, β) at $\gamma = 0.4$ ($\mathrm{Al} = 9, 10, \ldots, 14$, $\mathrm{Al}_1 = 11.5$, and $\mathrm{Al}_2 = 5.6$).

As follows from the expression for λ, the first approximation leads to wave damping and a decrease in the oscillation frequency by the value of the logarithmic decrement. For $\mathrm{Al} < \mathrm{Al}_1 = 11.5$, as Re_m decreases, the logarithmic decrement α has a maximum and then tends to zero as $\mathrm{Re}_m \to 0$ (undamped resonant oscillations). If $\mathrm{Al} > \mathrm{Al}_1$, a decrease in Re_m corresponds to an increase in the logarithmic decrement and a decrease in the oscillation frequency to zero, after which two aperiodic (nonoscillatory) motions of the fluid arise. Maximum amplitudes of the vertical displacements are reached on the fluid surface.

The second mode (*low-frequency electromagnetic surface waves*) (Fig. 2) has a zero limiting eigenvalue for $\mathrm{Re}_m \to \infty$ and the following asymptotic relation for the boundary layer:

$$\lambda = \frac{1 + i\sqrt{3}}{2} \sqrt[3]{\frac{(1 + (1+\gamma)(1 + \mathrm{Al}\,\gamma))^2}{\mathrm{Re}_m \sqrt{\mathrm{Al}}\,(1 + \mathrm{Al}\,(1+\gamma))^2}} + O\Big(\frac{1}{\mathrm{Re}_m^{2/3}}\Big).$$

In the second mode, the spectrum has properties similar to the properties of the spectrum in the first mode but with satisfaction of the inequality $\mathrm{Al} < \mathrm{Al}_1$. This complex behavior of the dispersion curves is due to the fact that at $\mathrm{Al} = \mathrm{Al}_1$ and $\mathrm{Re}_m = \mathrm{Re}_{m1}$ there is a double complex eigenvalue through which the separatrices dividing both modes pass. It is established that the behavior of the multiple mode is given by the law $(F_k^*(z) + tF_k(z)) \exp(-\lambda_k t)$, where $F_k(z)$ and $F_k^*(z)$ are the natural and associated functions. Maximum vertical displacements again occur on the fluid surface. A similar pattern was obtained in [8] for $\gamma = 0$.

Figure 3 shows the dispersion pattern for $\gamma = -0.8$ (the magnetic intensity decreases from the surface to the bottom) and fixed Alfvén numbers $\mathrm{Al} = 1.5, 1.75, \ldots, 3.0$. One can see the above-mentioned two oscillation modes, separated by the value $\mathrm{Al}_1 = 2.05$, and, in addition, there are unstable oscillations with growing amplitude for $\mathrm{Al} > \mathrm{Al}_2 = 3.15$ and small magnetic Reynolds numbers. Figure 4 shows the unstable region of the spectrum in greater detail. Here $\gamma = -0.4$, the Alfvén numbers are $\mathrm{Al} = 5.45, 5.5, \ldots, 5.8$, and instability begins at $\mathrm{Al} > \mathrm{Al}_2 = 5.6$.

Table 1 gives the dependences of the critical Alfvén numbers $\mathrm{Al}_i(\gamma)$ ($i = 1$ corresponds to the double complex eigenvalue and $i = 2$ corresponds to the onset of instability).

For applications, the asymptotic behavior of all modes at $\mathrm{Re}_m \to 0$ [1] is of interest. Constructing this asymptotics is a nontrivial mathematical problem. Therefore, in the present work, we only give the final formula

$$\lambda = i\sqrt{\frac{1}{\mathrm{Al}} + \frac{\gamma(\gamma+3)}{6}} + O(\mathrm{Re}_m),$$

which is valid for a positive radicand. Otherwise, instability occurs. Calculations of the dispersion patterns

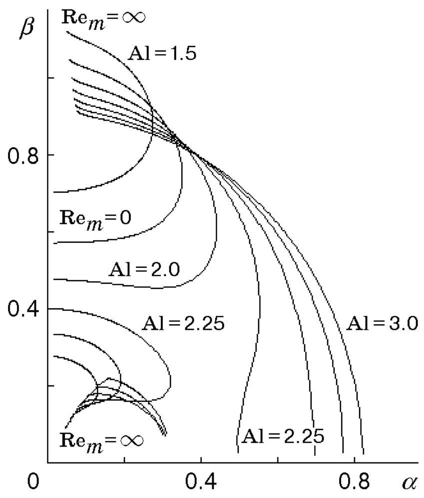

Fig. 3

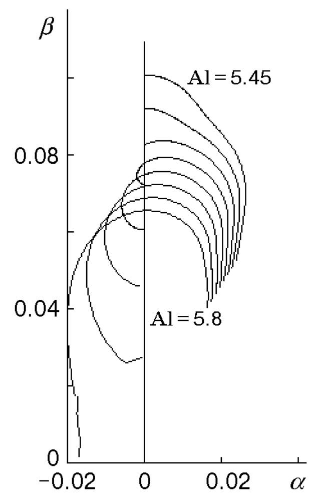

Fig. 4

Fig. 3. Dispersion curves in the plane (α, β) for $\gamma = -0.8$ ($\mathrm{Al} = 1.5, 1.75, \ldots, 3.0$, $\mathrm{Al}_1 = 2.1$, and $\mathrm{Al}_2 = 3.1$).

Fig. 4. Unstable modes for $\gamma = -0.4$ ($\mathrm{Al} = 5.45, 5.5, \ldots, 5.8$, $\mathrm{Al}_1 = 2.9$, and $\mathrm{Al}_2 = 5.6$).

TABLE 1

γ	Al_1	Al_2
−0.8	2.05	3.15
−0.6	2.43	3.80
−0.4	2.95	5.60
−0.2	3.68	10.60
0	5.15	—
0.2	6.77	—
0.4	11.10	—
0.6	32.50	—

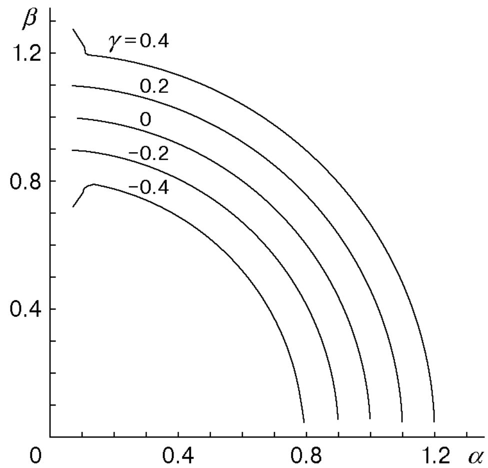

Fig. 5. Dispersion curves in the plane (α, β) for Alfvén internal dissipative waves ($\gamma = -0.4$, -0.2, 0, 0.2, and 0.4).

given in Figs. 3 and 4 and in Table 1 confirm this result, whose possibility was discussed in deriving the energy-balance equation.

There is one more type of oscillations, called *Alfvén internal dissipative waves* (Fig. 5). In this case, the dispersion curves depend weakly on Al and practically merge. For the present value of Al, we have a finite number of oscillatory modes that correspond to different ranges of Re_m. At the points M_k located on the material axis, the oscillatory motion of the fluid is converted to two aperiodic motions (double real eigenvalues). They correspond to a monotonically increasing sequence of numbers $\mathrm{Re}_{m,k} \to \infty$. For the kth mode, we have $\mathrm{Re}_m > \mathrm{Re}_{m,k}$, but as $\mathrm{Re}_m \to \infty$, all modes have one limiting point [the second, smaller, root of the dispersion equation of the degenerate problem (22)]. The neighborhood of the limiting point contains a finite number of eigenvalues that can become as large as one likes with increase in Re_m. For rather large Reynolds numbers, at depth $z = (1 - \mathrm{Im}\,(\lambda))/\gamma$ there is a turning point of the differential operator. In the neighborhood of this point, the velocity field undergoes a sharp change which degenerates into a tangential discontinuity of the second kind that takes place in an infinitely conducting fluid. All oscillation modes of the fluid are of the character of internal waves because their maximum amplitudes are located inside the layer. These waves are adequately described by a combination of the approximation of "a rigid boundary" $[W(0) = 0]$ and the Boussinesq approximation ($|\gamma| \ll 1$ and $H = \mathrm{const}$), and this makes it possible to obtain all results in analytical form.

REFERENCES

1. I. M. Kirko, *Liquid Metal in an Electromagnetic Field* [in Russian], Énergiya, Moscow–Leningrad (1964).
2. Yu. P. Ladikov and V. F. Tkachenko, *Magnetohydrodynamic Instabilities in Metallurgical Processes* [in Russian], Nauka, Moscow (1983).
3. V. A. Barinov and N. G. Taktarov, *Mathematical Modeling of Magnetohydrodynamic Surface Waves* [in Russian], Mordov. Univ., Saransk (1991).
4. L. V. Cherkesov, *Hydrodynamics of Surface and Internal Waves* [in Russian], Naukova Dumka (1976).
5. V. E. Fertman, *Magnetic Fluids* [in Russian], Vysshaya Shkola, Minsk (1988).
6. J. Hall and J. Watt, *Modern Numerical Methods for Ordinary Differential Equations*, Clarendon Press, Oxford (1976).
7. S. A. Lomov, *Introduction to the General Theory of Singular Perturbations* [in Russian], Nauka, Moscow (1981).
8. A. I. Zadorozhnyi, "On the spectrum of one integrodifferential operator that arises in the theory of long MHD waves," in: *Integrodifferential Operators and Their Applications* (collected scientific papers), No. 2, Izd. Rostov. Tekh. Univ., Rostov-on-Don (1997), pp. 79–88.